Simulation of Sound Propagation in an Axisymmetric Duct

Report for project in TT8303 Numerical Acoustics

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Abstract

A method for calculating the propagation of higher modes in a duct of varying cross-section is implemented using Matlab. The duct is approximated as a series of cylindrical sections. For verification of the method, only a single discontinuity is investigated, and the results are compared to results obtained from simulations using the Boundary Elements Method (BEM) and the Boundary Element Rayleigh Integral Method (BERIM). A method for decomposing a wave front in a cylindrical tube into its modal components is described, and applied to the BEM results. Ordinary BEM has problems satisfying the totally absorbing boundary conditions at the end of the tube. BERIM, which imposes a radiating boundary condition at the end of the tube, compares more favorably with the modal propagation method.

1 Introduction

Ducts of varying cross-section are used in many applications. Two common examples are loudspeaker horns and musical instrument horns. At low frequencies, these ducts can often be analyzed by the so-called horn equation, see for instance Webster [1]. The equation is applicable as long as there is only axial variations of pressure along the duct. However, when the horn cross-section becomes large compared to the wavelength, the waves in the duct will also exhibit variations in the two other dimensions. For a circular duct, the variations will be both radial and angular. This is due to the introduction of higher modes [2]. Some of these modes will propagate through the duct and will be radiated into air, others will decay exponentially. But both kinds of modes will take their energy from the fundamental mode.

The presence of higher modes will alter both the acoustical impedance seen from the throat (input) end of the duct, the internal pressure distribution, and the radiation pattern from the mouth (output) end. Accurate calculation of the loading and radiating properties of horns can only be performed if this taken into account.

This report describes the implementation of a method for calculating the propagation of modes along a duct of varying cross section. The duct is approximated by a number of steps, using a series of cylindrical sections. This approach was probably first implemented by Alfredson [3], who used an iterative approach. The method that is implemented here, is the one described by Kemp [4], which is similar to the one described by Pagneux [5]. This method includes both means for propagating modes along a uniform cylinder, and across a discontinuity.

To verify the method, a test case consisting of a single discontinuity is simulated using both the modal propagation method (MPM), the Boundary Elements Method (BEM), and the Boundary Element Rayleigh Integral Method (BERIM). To compare the levels of the modes in each model, one can decompose the pressure distributions calculated by BEM into normal modes. A method for performing this decomposition is shown and evaluated.
2 Theory

The theory section contains one part on the theory of modal propagation in ducts, and one part on the BEM and the BERIM. Since BEM/BERIM is not the main topic of this project, the theory sections on these methods are quite cursory.

2.1 Modal propagation in ducts

The pressure and velocity in a uniform duct can be expressed as a weighted sum of allowable modes [6, 7]. In a cylindrical duct, there will be $n$ nodal circles, and $l$ nodal diameters, and $(n, l) = (0, 0)$ represents the plane wave mode. For an axisymmetric duct with axisymmetric excitation, only axisymmetric modes (nodal circles) will exist.

The Helmholtz equation for cylindrical coordinates is

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} = -k^2 p$$

where $r$ and $\theta$ gives the polar coordinates in the cut plane of the duct, and $z$ is the coordinate along the axis of the duct. The solution can be expressed in the form

$$p(r, \theta, z) = \sum_{n=0}^{\infty} P_n(z) \psi_n(r, \theta)$$

where $P_n$ is the pressure profile along the tube, and $\psi_n$ is the pressure profile in the $(r, \theta)$ plane. If we substitute $p$ from eq. (2) into eq. (1) and divide by $p[4]$, we get

$$\frac{1}{\psi_n} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi_n + \frac{1}{P_n} \frac{\partial^2 P_n}{\partial z^2} = -k^2$$

where $k = \omega/c$ is the free-space wavenumber. Using separation of variables, we get

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi_n = -\alpha_n^2 \psi_n$$

From this follows that

$$k^2 = k_n^2 + \alpha_n^2$$

$\alpha_n$ is the eigenvalue of the $n$th mode, the allowed wave number in the radial direction, while $k_n$ is the wave number in the axial direction.

For propagation in the $z$ direction, we get from eq. (5) that

$$P_n(z) = A_n e^{-ik_n z} + B_n e^{ik_n z}$$

We can see that since

$$k_n = \pm \sqrt{k^2 - \alpha_n^2}$$

the axial wave number will for for certain values of $\alpha_n^2$ be imaginary, and the propagation in $z$ direction will be evanescent (exponentially damped). The frequency when $k_n$ becomes real is called the cut-off (or some times cut-on) frequency of the corresponding mode, $k_c = \alpha_n$.

The solution of eq. (4) is a bit more involved. We can simplify the solution by assuming axial symmetry, in this case the term depending on $\theta$ will disappear. We are then left with

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \psi_n = -\alpha_n^2 \psi_n$$

which can be rewritten as Bessel’s equation of order zero. The solution of this equation is

$$\psi_n(r) = c_1 J_0(\alpha_n r) + c_2 Y_0(\alpha_n r)$$

where $J_0$ and $Y_0$ are Bessel functions. Since $Y_0$ is singular when its argument is zero, $c_2$ must be zero.

For a rigid-walled duct of radius $R$ we have that

$$\frac{\partial \psi_n}{\partial r} = 0, \quad r = R$$

For this to be the case, $\alpha_n$ must be so that

$$\alpha_n J_0'(\alpha_n R) = 0$$

From the relation

$$\frac{d}{d(x)} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x)$$

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we get that $J'_0 = J_1$. If $\gamma_n$ are the successive zeros of $J_1$, we get that
\[ \alpha_n = \frac{\gamma_n}{R} \] (14)
and $\psi_n$ can be expressed as (normalized to the amplitude at $r = R$):
\[ \psi_n = \frac{J_0(\gamma_n r/R)}{J_0(\gamma_n)} \] (15)
A plot of the first 10 mode shapes is shown in fig. 1. In this figure, the mode shapes are not normalized. Mode 0 is the plane wave mode, which can clearly be seen since it does not vary with the radius. The other modes have increasingly complex shapes, and all modes ends at the duct wall with zero slope, a requirement for the hard wall boundary condition. All possible wave front shapes in the duct will be a weighted sum of these mode shapes.

### 2.1.1 Propagation along a duct

For a wave propagating along a duct, we have
\[ P_n(z) = A_n e^{-ik_n z} + B_n e^{ik_n z} \] (16)

From eq. (8) we have that the wave number of each mode is
\[ k_n = \pm \sqrt{k^2 - \left(\frac{\gamma_n}{R}\right)^2} \] (17)
We must choose the right signs of the square root to get the correct behavior of the waves. When the mode is propagating, we must have the normal equation of propagation, so we must take the positive root. When the mode is in cutoff, the modal wave number is purely imaginary, and the wave should be exponentially damped with distance. We must therefore take the negative root. In summary
\[ k_n = \begin{cases} -\sqrt{k^2 - \left(\frac{\gamma_n}{R}\right)^2} & : k < \frac{\gamma_n}{R} \\ \sqrt{k^2 - \left(\frac{\gamma_n}{R}\right)^2} & : k > \frac{\gamma_n}{R} \end{cases} \] (18)
2.1.2 Propagation across a discontinuity

The geometry is illustrated in fig. 2. When the wave propagates across the discontinuity, there must be continuity of pressure and velocity right before (at point 1) and right after the discontinuity (at point 2). But the $n$th mode in duct 1 will not match the $n$th mode in duct 2, so the pressure field in duct 2 must be made up of a new sum of modes. Every mode in duct 1 excites a series of modes in duct 2, this is known as modal coupling or mode conversion. Kemp [4] derives expressions for this coupling.

If $\vec{P}^{(1)}$ is the vector of modal pressure amplitudes at point 1, and $\vec{P}^{(2)}$ is the vector of modal pressure amplitudes at point 2, and $R_1 < R_2$, the vectors are related by a matrix $F$ so that

$$\vec{P}^{(1)} = F \vec{P}^{(2)} \quad (19)$$

were the matrix elements are defined as

$$F_{nm} = \frac{1}{S_1} \int_{S_1} \psi^{(1)}_n(\vec{r}) \psi^{(2)}_m(\vec{r}) dS \quad (20)$$

If $R_1 > R_2$, the relation is

$$\vec{P}^{(1)} = V \vec{P}^{(2)} \quad (21)$$

were the matrix elements are defined as

$$V_{nm} = \frac{1}{S_2} \int_{S_2} \psi^{(2)}_n(\vec{r}) \psi^{(1)}_m(\vec{r}) dS \quad (22)$$

For an axisymmetric geometry, setting $\beta = R_1/R_2$, the matrices turn out to be

$$F_{nm}(\beta) = \frac{2\beta \gamma_m J_1(\beta \gamma_m)}{(\beta^2 \gamma^2_m - \gamma^2_n) J_0(\gamma_m)} \quad (23)$$

and

$$V_{nm}(\beta) = F_{nm}(1/\beta) \quad (24)$$

where $F_{(0,0)} = V_{(0,0)} = 1$.

For a complete derivation of this equation, see Kemp [4, appendix B].

Volume velocity can be propagated across the discontinuity as

$$\vec{U}^{(2)} = F^T \vec{U}^{(1)}, \quad S_1 < S_2 \quad (25)$$

$$\vec{U}^{(1)} = V^T \vec{U}^{(2)}, \quad S_1 > S_2 \quad (26)$$

2.1.3 Modal radiation impedance

In many practical cases, the duct (horn) is designed to radiate into the air. The duct therefore has to be terminated with a proper radiation impedance. The radiation impedance for a piston of area $S$, and with uniform velocity distribution, is often used, and the expression is well known:

$$Z = \frac{\rho c}{S} \left[ 1 - \frac{J_1(2ka)}{ka} + j \frac{H_1(2ka)}{ka} \right] \quad (27)$$

This is only the radiation impedance for the plane wave mode, however, and we need to find the radiation impedance for all the modes. This is described by Zorunski [8] and Kemp [4], and can be found from the integral

$$Z_{nm} = \frac{j\omega \rho}{2\pi S^2} \left( \frac{R}{R_0} \right)^2 \int_0^{2\pi} d\theta \int_0^\infty rdr \int_0^\infty r_0 dr_0 F \quad (28)$$

where

$$F = \psi_m(r_0, \theta_0) \psi_n(r, \theta) e^{-jkh} \quad (29)$$

$$h = [\tau^2 + r_0^2 - 2r_0 \cos(\theta - \theta_0)]^{\frac{1}{2}} \quad (30)$$

After some extensive manipulation of the integrals, the result, as given by the references [8, 4], is reduced to two single integrals:

$$Z_{nm} = \frac{\rho c}{S} \left( \frac{\pi/2}{\sin \phi} \right) D_n(\sin \phi) D_m(\sin \phi) d\phi + \frac{j\rho c}{S} \left( \frac{\sinh \xi}{\cosh \xi} \right) D_n(\cosh \xi) D_m(\cosh \xi) d\xi \quad (31)$$

where

$$D_n = -\sqrt{2\pi} \frac{J_1(\tau kR)}{\left( \frac{\tau}{kR} \right)^2 - \tau^2} \quad (32)$$

These integrals are then numerically integrated, except for the $(0,0)$ mode, where the exact analytical expression can be used to improve speed.

In the actual implementation, the matlab function quad was used. According to Kemp, the integrand
of the reactance function decays exponentially quite fast to about $10^{-6}$ around $\xi = 10$, so the integration can be terminated here without much error.

The impedance at the diagonal of the resulting matrix is the radiation impedance of each mode. The other entries in the matrix describe the coupling between the various modes.

2.1.4 Projection of impedance and volume velocity

The pressure and volume velocity is related through the modal impedance. We know the impedance at the end of the duct. To calculate the pressure field inside the duct, we first need to know the impedances through the duct. Then we can apply, for instance, a plane wave of constant velocity at one end, and propagate this velocity through the duct using the equations given in sections 2.1.1 and 2.1.2. The pressure is then found by multiplying the velocity at the point in question by the impedance at that point.

Kemp gives the equations for propagating the impedance from duct 2 to duct 1 as

$$Z^{(1)} = FZ^{(2)}F^T, S_1 < S_2$$  \hspace{1cm} (33)

$$Z^{(1)} = V^{-1}Z^{(2)}(V^T)^{-1}, S_1 > S_2$$  \hspace{1cm} (34)

For propagation along a uniform duct, where $Z^{(0)}$ is the impedance at the input of the duct, due to an impedance $Z^{(1)}$ at the output, the relation is

$$Z^{(0)} = \left( Z^{(1)} + jD_3Z_c \right) \left( jD_3Z_c^{-1}Z^{(1)} + I \right)^{-1}$$  \hspace{1cm} (35)

The extra matrices are defined as follows:

$$D_3(n,m) = \begin{cases} \tan(k_n d) & : n = m \\ 0 & : n \neq m \end{cases}$$  \hspace{1cm} (36)

$$Z_c(n,m) = \begin{cases} kpc/k_n S & : n = m \\ 0 & : n \neq m \end{cases}$$  \hspace{1cm} (37)

where $d$ is the length of the duct as in fig. 2, and $S$ is the cross-sectional area.

Volume velocity is propagated along the duct as

$$\vec{U}^{(1)} = \left( -D_2Z_c^{-1} \left( Z^{(0)} - Z_c \right) + E \right) \vec{U}^{(0)}$$  \hspace{1cm} (38)

where the extra matrices are:

$$D_2(n,m) = \begin{cases} j\sin(k_n d) & : n = m \\ 0 & : n \neq m \end{cases}$$  \hspace{1cm} (39)

$$E(n,m) = \begin{cases} e^{-jk_n d} & : n = m \\ 0 & : n \neq m \end{cases}$$  \hspace{1cm} (40)

2.2 Boundary Elements Method

Since BEM is not the main topic of this report, only a brief overview of the method will be given here. It is taken from Kristiansen [9], and also from Kirkup [10], who has also developed the codes that have been used.

We want to solve the Helmholtz equation in velocity potential $\varphi$ at the point $p$:

$$\nabla^2 \varphi(p) + k^2 \varphi(p) = 0$$  \hspace{1cm} (41)

By using Green’s second identity, we can reduce what would require a volume integral over the domain $D$, to an integral over only the boundary $\Gamma$. The result is the Kirchhoff-Helmholtz integral theorem:

$$\varphi(p) = \frac{1}{4\pi} \int_{\Gamma} \left( e^{-ikr} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial e^{-ikr}}{\partial n} \right) d\Gamma$$  \hspace{1cm} (42)

(Note that velocity potential is used instead of pressure, as in [9]). Discretizing this equation, we obtain

$$\varphi(p) \approx \frac{1}{4\pi} \sum_j \left[ \frac{\partial \varphi_j}{\partial n} \int_{\Gamma_j} e^{-ikr} \frac{d\Gamma}{r} - \varphi_j \int_{\Gamma_j} \frac{\partial e^{-ikr}}{\partial n} \frac{d\Gamma}{r} \right] (p \in D)$$  \hspace{1cm} (43)

We need to know both the velocity potential and velocity on the surface. If we place $p$ on the surface, we can evaluate the potential here. The equation in
Figure 3: Normalized modal radiation impedance.

this case becomes

\[ \frac{1}{4\pi} \sum_j \left( \varphi_j \int_{\Gamma_j} \frac{\partial}{\partial n} \frac{e^{-ikr}}{r} d\Gamma + \frac{1}{2} \varphi_j \right) = \frac{1}{4\pi} \sum_j \frac{\partial\varphi_j}{\partial n} \int_{\Gamma_j} \frac{e^{-ikr}}{r} d\Gamma \]  

(44)

which can be rewritten as a matrix equation that can be solved on a computer. Note that in BEM, the matrices are full, and for large problems, BEM can be less efficient than the finite elements method (FEM). However, for infinite domain problems, as acoustic radiation problems often are, the advantage in discretizing only the surface can be huge.

2.2.1 Operator notation

The integral equations can be written using operator notation [10], which is a compact way to write operations. For the BEM method used, we have four operators, defined as follows:

\[ \{L_\kappa \zeta\}_\Gamma (p) = \int_{\Gamma} G_k(p, q) \zeta(q) dS_q \]  

(45)

\[ \{M_\kappa \zeta\}_\Gamma (p) = \int_{\Gamma} \frac{\partial G_k(p, q)}{\partial n_q} \zeta(q) dS_q \]  

(46)

\[ \{M_k^u \zeta\}_\Gamma (p; u_p) = \frac{\partial}{\partial u_p} \int_{\Gamma} G_k(p, q) \zeta(q) dS_q \]  

(47)

\[ \{N_k \zeta\}_\Gamma (p; u_p) = \frac{\partial}{\partial u_p} \int_{\Gamma} \frac{\partial G_k(p, q)}{\partial n_q} \zeta(q) dS_q \]  

(48)

where \( \Gamma \) is a (partial) boundary, \( n_q \) is the unit normal vector to \( \Gamma \), \( u_p \) is a unit vector, \( \zeta(q) \) is a function defined for \( q \in \Gamma \), and \( G_k(p, q) \) is the free space Green’s function,

\[ G_k(p, q) = \frac{1}{4\pi r} e^{-ikr} \]  

(49)

with \( r = |p - q| \).
2.2.2 Interior BEM

Using the operator notation, the Helmholtz equation can be formulated as

$$\{M_k \varphi\}_\Gamma (p) + \varphi(p) = \{L_k v\}_\Gamma (p) \quad (p \in D) \quad (50)$$

$$\{M_k \varphi\}_\Gamma (p) + \frac{1}{2} \varphi(p) = \{L_k v\}_\Gamma (p) \quad (p \in \Gamma) \quad (51)$$

for \(p\) in the domain \(D\) or on the surface \(\Gamma\). Note that this formulation is eqs (43) and (44) rewritten in operator notation.

This formulation is not suitable for all boundary conditions, so a combination of this formulation and another formulation are combined to give

$$\left\{ \left( M_k + \frac{1}{2} I + \mu N_k \right) \varphi \right\}_\Gamma (p; n_p) + \frac{1}{2} \varphi(p) = \left\{ \left( L_k + \mu \left( M_k - \frac{1}{2} I \right) \right) v \right\}_\Gamma (p; n_p) \quad (52)$$

where \(\mu\) is a coupling parameter, set to \(\frac{1}{1+i}\) in typical applications. See Kirkup [10] for a discussion. This formulation can be used for the general boundary condition

$$\alpha(p) \varphi(p) + \beta(p) v(p) = f(p) \quad (53)$$

This equation describes the relation between the velocity potential \(\varphi\) and the velocity \(v\) at the boundary in point \(p\). The possibility to use a general boundary condition is a requirement for being able to for instance create an absorbing wall with a \(\rho c\) impedance.

The above method is implemented in the axisymmetric routine AIBEMA, which is used to calculate the internal field of the perfectly terminated stepped tube used in this project.

2.3 Boundary Elements Rayleigh Integral Method (BERIM)

For certain problems, for instance cavities coupled to an infinite baffle, the traditional exterior BEM can be inefficient. A method to couple the interior boundary integral equation in the cavity with the Rayleigh integral in the opening, has been described by Kirkup [11].

To couple the interior and exterior, the cavity opening is replaced by a surface \(\Pi\). Using operator notation, the direct formulation for the points in the cavity will be

$$\{M_k \varphi\}_{\Gamma \cup \Pi} (p) + \varphi(p) = \{L_k v\}_{\Gamma \cup \Pi} (p) \quad (p \in D) \quad (54)$$

and the Rayleigh integral for the points on \(\Pi\) and in the domain \(E\) in front of the cavity

$$\varphi(p) = -2 \{L_k v\}_{\Pi} (p) \quad (p \in \Pi \cup E) \quad (55)$$

For the points on the surfaces,

$$\{M_k \varphi\}_{\Gamma \cup \Pi} (p) + \frac{1}{2} \varphi(p) = \{L_k v\}_{\Gamma \cup \Pi} (p) \quad (p \in \Gamma \cup \Pi) \quad (56)$$

The inner surface can be separated from the mouth:

$$\{M_k \varphi\}_\Gamma (p) + \{M_k \varphi\}_\Pi (p) + \varphi(p) = \{L_k v\}_\Gamma (p) + \{L_k v\}_\Pi (p) \quad (p \in D) \quad (57)$$

$$\{M_k \varphi\}_\Gamma (p) + \frac{1}{2} \varphi(p) = \{L_k v\}_\Gamma (p) + \{L_k v\}_\Pi (p) \quad (p \in \Gamma \cup \Pi) \quad (58)$$

This can be reformulated into a linear system of equations, and the velocity and velocity potential on the surfaces, and the velocity potential in the domains can be found.
3 Modal decomposition

From (2) we know that the pressure in a given position in the tube can be expressed as a sum of modal eigenfunctions. For a given position $z$, we have

$$p(r, z) = \sum_{n=0}^{\infty} P_n(z) \psi_n(r)$$

where $P_n$ can be thought of as the amplitude of mode $n$. We typically want to decompose the pressure distribution in a given plane into its normal modes, i.e., find the values $P_n$. Modes of high order will have high cut-off frequencies, and many of them will be evanescent at the frequency of interest. They may still have quite high amplitudes close to points where they are generated (like discontinuities), and a certain amount of them should be retained.

For numerical implementation, we do of course have to limit the number of modes, and to discretize the pressure distribution in the plane of interest. Eq. (59) can then be rewritten as

$$p_i = \sum_{j=0}^{M} P_j \psi_{ij}$$

where the index $i$ runs through the sample points, and the index $j$ runs through the modes. This is a simplified version of the method used by Morgans [12], who also included the angular modes. This equation can be written in vector form as

$$\vec{p} = \Psi \vec{P}$$

where $\Psi$ is a (not necessarily square) $N \times M$ matrix containing the modal shapes as a function of $r$ as rows, and as function of mode number as columns. That is, plotting column $j$ of the $\Psi$-matrix gives the shape of the mode $j$ as a function of $r$. $\Psi$ is calculated from (15) for the $N$ pressure sample points in the duct, usually placed linearly from the center out to the duct wall. $\vec{P}$ is a column vector of the $M$ modal amplitudes, and $\vec{p}$ is a column vector containing the pressure at $N$ points in the duct. We can now find the modal amplitudes as

$$\vec{P} = \Psi^{-1} \vec{p}$$

Since for most cases, $N \neq M$, the Matlab function pinv (pseudo-inverse) was used [12]. This function is based on singular value decomposition (SVD) [13].

3.1 BEM simulation

To have a benchmark to test the modal propagation code against, a test problem was simulated using interior BEM. The code used is described in detail in [10], and is freely available on the web [14] where also the book can be downloaded. The boundary used is shown in fig. 5. The boundary consists of truncated conical shells, and the mesh size is selected so that there will be at least 12 elements per wavelength at 8kHz, which is the highest frequency used. The small end is driven by a rigid piston with the velocity of Im/s, and the other end is terminated by a surface that has been given the characteristic impedance of air, $pc$.

The configuration consists of a 10cm long tube of 1cm radius that is coupled to another 10cm long tube of 5cm radius. There is thus a quite large discontinuity at $z = 10cm$, which is expected to generate many higher modes.

Field points are placed at lines perpendicular to the axis at regular intervals. The complex pressure and coordinates for each field point was exported as a text file, and a Matlab function was written to read the file into a data structure where the field point groups could be identified.

3.2 Convergence

Before the modal decomposition results could be used as a basis for comparison, the number of field points required for a given number of modes had to
be determined. If the spatial sampling of the mode shapes is too coarse, proper decomposition may not be possible.

The point right after the discontinuity, \( z = 0.11m \), was selected as a basis, and several lines of field points were placed at this point, with resolutions of 10, 20, 30, 40, 50, 65, 80, 105, 130 and 150 points. Modal decomposition was then performed with various number of modes.

Figure 6 on the following page\(^1\) shows the modal amplitudes for 40 decomposed modes. Figures 7 and 8 (page 11) show the modal amplitudes for 80 and 150 modes, respectively.

The mode levels are not plotted when the mode number is greater than 1.5 times the number of field points. Another thing to note is that when the number of field points is less than the number of modes, we have an underdetermined system of equations, and while \( \text{pinv} \) gives an answer, the results are dubious. While it still seems like the levels of the dominant modes are fairly accurate, errors introduced from the higher modes may be a problem.

It can be seen from the plots that the modal amplitudes converge when the number of field points exceed the number of modes retained. It can also be seen that the lower modes do not need as many field points to converge, as the higher modes do. This is probably due to both the accuracy of the representation of the mode shapes, and the relative amplitudes of the modes.

It can be seen that the levels of most of the higher modes level out at approximately the same value. In the interest of checking if this was due to the resolution of the mesh, another simulation using 48 elements per wavelength at 8 kHz was performed. The results are shown in fig. 9 on page 11. It can be seen that the levels of the higher modes is around 10 dB lower.

In conclusion, the number of field points should exceed the number of modes it is desired to calculate. One could also have assumed this from fig 1, when looking at the mode shapes. The ninth mode has

\(^1\) The modes have not been identified by a legend in these plots, due to the large number of modes. Also, the purpose of the figures is to show where the modes converge to a nearly asymptotic level.
Figure 6: Modal amplitudes as function of number of field points, 40 modes

Modal amplitudes as function of Ncoord, f = 2971.9886 Hz

Figure 7: Modal amplitudes as function of number of field points, 80 modes

Modal amplitudes as function of Ncoord, f = 2971.9886 Hz
Modal amplitudes as function of Ncoord, $f = 2971.9886\text{Hz}$

Figure 8: Modal amplitudes as function of number of field points, 150 modes

Modal amplitudes as function of Ncoord, $f = 2971.9886\text{Hz}$

Figure 9: Modal amplitudes as function of number of field points, 80 modes, high resolution mesh
nearly 5 “cycles”. Applying the sampling theorem, we should have at least two samples per cycle of a periodic signal, and since the number of “cycles” is about half the mode number, the mode number will give the lower limit on the number of sampling points.

When few modes are calculated, the level of the highest calculated modes will not be correct. The reason for this is that the \texttt{pinv} function will give an answer in the least-squares sense \cite{15}, and this will distribute the values of the missing modes across the existing modes. This can be seen in figs. 10 and 11, where the values of the modes are higher when there is a low number of modes used in total in the decomposition. In retrospect, this should have been taken into account when calculating the plots in figures 6–9, so that no more modes than the number of field points were calculated at any point.

It became clear that setting the impedance of one wall to \( \rho c \) will not correspond to terminating the duct by \( Z_c \) from eq. (65), as there is no way to set the modal impedances in BEM. To check if this was a factor, the wave front pressure distribution at \( z = 0.11 \text{m} \) was observed while varying the length of the second tube. It was noted that the pressure varied quite a bit, both in value and distribution, especially at the higher frequencies, see fig. 12. This indicates that the higher modes are not terminated equally depending on the length of the second tube.

The convergence tests were done without normalization of the \( \Psi \)-matrix.

### 3.3 Example decomposition results

A few examples of modal decomposition is shown in this section. The planes where the field has been decomposed is at \( z = 0.09 \text{m} \), and at \( z = 0.11 \text{m} \), i.e. 10mm before and after the discontinuity. Figure 13 shows the pressure distribution across the tube from 25 field points, and the reconstructed wavefront. The wavefront has been reconstructed from the amplitudes of the 10 retained modes (including the plane wave mode) in the decomposition. Here we see one problem with BEM: the wavefront is not perpendicular to the wall, but drops sharply at the field point closest to the wall. The attempted re-
construction shows ringing analogous to the Gibb’s phenomenon. From figure 14 we see that the plane wave mode is dominant, but the other modes levels cut about 30dB below this. This is probably not quite correct, and can be caused both by the problems near the wall, and the resolution of the boundary, as discussed in section 3.2.

After the discontinuity, the pressure distribution looks like what is shown in fig. 15. 25 modes, including the plane wave mode, are retained, and the reconstruction is much better. But we have the small drop of pressure near the edge also in this case. The dominant mode is the first (n=1) mode, and from the fifth mode and upwards, the level is quite stable, see fig. 16 on the following page. Here, as in the previous case, the reason may be the problems near the wall, and the boundary resolution.
4 Modal propagation

The method is implemented fairly directly from the theory described in section 2.1. For each frequency, the modal impedance at the terminating end of the duct, either $Z_c$ (for perfect termination) or the modal radiation impedance from eq (31) is calculated, and propagated down to the driven end of the duct using eq. (35) and (33).

The starting volume velocity is then set as a vector of modal velocities where only the plane wave mode (mode 0) is used. This velocity is then propagated down the tube using eq (38) and (25). At the point where it is desired to calculate the pressure vector, the volume velocity is multiplied by the impedance matrix to give

$$\vec{P} = Z \vec{U} \quad (63)$$

The pressure distribution can then be reconstructed as

$$p(r) = \psi \vec{P} \quad (64)$$

4.1 Implementation details

The method is implemented in Matlab. The geometry of the tube is given as a short list of coordinates, setting the ends and the discontinuities. This geometry is then refined to reduce the element lengths to below a certain maximum value, $\Delta z$. Other input values are the frequency limits, the number of frequencies, and the number of modes used. For tubes terminated in a baffle, the modal radiation impedance is then calculated, while for plane wave terminated tubes, $Z_c$ is used. This impedance is then propagated back from the mouth to the throat end, and the impedance matrix for each point is stored. This is done in a double loop, the outer loop going through the list of frequencies, and the inner loop going through the tube elements from end to end. When these loops are completed, the array of modal impedances at each point can be used to calculate the pressure at a given frequency and point in the tube, by propagating the velocity to that point.

This is, as described above, done by propagating the volume velocity from the input end to the point where it is desired to calculate the pressure. In the current implementation, this is done for a single frequency, but this can easily be changed.

It turned out that velocity could not be propagated over large distances without great numerical errors. These errors comes from the evanescent modes, which due to their small amplitudes at certain points, will give nearly singular matrices. As more modes were included, the modal amplitudes grew until they created numerical overflow in Matlab. After personal communication with Kemp, it was tried to subdivide each tube into shorter elements, and increasing the number of elements until convergence was reached. This solved the problem.

Pressure distribution 10mm after the discontinuity is is shown in fig. 17 for various values of element lengths, $\Delta z$. 10 modes are included. Compare the results to the curves given in fig. 15. The shapes (for small $\Delta z$) are very similar, but the amplitude is larger than in the BEM calculations.

Another issue that turned up, was how to best implement eqs. (35) and (38). The equations includes inversion of matrices, and some inversion methods
Pressure distribution at \( f = 2972\,\text{Hz}, z = 0.11 \) turned out to be better suited to the task than others. For instance, the `inv()` method often produced warnings and large mode amplitudes\(^2\). It was also found desirable to compute \( Z_c^{-1} \) directly, without the use of matrix inversion functions, as

\[
Z_c^{-1}(n, m) = \begin{cases} 
  k_n S / kpc & : n = m \\
  0 & : n \neq m 
\end{cases} \quad (65)
\]

### 4.2 Calculation of the pressure response

Calculation of the pressure response of the horn in the far field is also necessary if the model is to be used for directivity predictions. One way to do this, for a radiator in an infinite baffle, is to use the Rayleigh integral [16, 17], mapping the velocity profile in the mouth of the horn to far field pressure response. The velocity in the mouth opening is found from

\[
u(r, z) = \frac{1}{S} \sum_{n=0}^{N} U_n(z) \psi_n(r). \quad (66)
\]

The Rayleigh integral is given as

\[
p(r) = \frac{ikpc}{2\pi} \int_S u(S) \frac{e^{-ikr}}{r} dS \quad (67)
\]

where \( u(S) \) is the velocity of a region of the radiator, and \( S \) is the surface.

This equation was implemented using simple numerical integration. The velocity in the case in question is constant in the angular direction, but we still need to integrate in both radial and angular direction. The number of integration points in the radial direction is given by the resolution of the wave front reconstruction. The angular resolution is calculated based on the ratio of the circumference to the central radius of the ring.

The results from the routine was compared to the analytical solution for the far field of a piston of uniform velocity, and was found to agree well.

### 4.3 Perfectly terminated tube

The configuration used for these simulations was the same as was used for the BEM simulations: two tubes of 10 cm length, the first with radius 1 cm and the second with radius 5 cm. The end was terminated by \( Z_c \) (eq (37)).

Calculating the wave front pressure distribution using this method, and using 10 modes, gives, for \( z = 0.09\text{m} \), the results shown in figures 18 and 19. The axis in fig. 18 is scaled to be approximately equal to that used in fig. 13, and it can be seen that the wave front is much more plane in this case. The higher modes are also lower in level.

### 5 Comparison with BEM and BERIM

A series of tests were run to compare the results of the modal propagation method (MPM) to the results of BEM and BERIM. Both input impedance and pressure distributions were compared.
5.1 Impedance calculations

The input acoustical impedance for a duct terminated with a plane wave termination is shown in fig. 20 on the next page. Only 4 modes (plane wave and 3 higher modes) are included in the modal model. The impedance shown is the $(0,0)$ impedance, corresponding to the average radiation impedance over the surface. The BEM model had 12 elements per wavelength at 8kHz. The modal model had an average element length of 0.1mm. The modal model used about 50 seconds, and the BEM model used 48 minutes to calculate this plot.

The agreement is quite good up to about 4.5kHz, above this frequency BEM has higher peaks and slightly different resonance frequencies. It is not unlikely that this is a result of the imperfect termination of the higher modes in the BEM model.

The input acoustical impedance for a duct terminated in an infinite baffle is shown in fig. 21 on the following page. For the MPM, the plane wave mode and 3 higher modes are included. Both models had the same mesh/element lengths as for the plane wave termination. Even with this few modes included, the agreement between the methods is very good, and even small features are in agreement. Since we here have a radiation condition, it is likely that there is a better agreement between BEM/IM and the modal model for the conditions at the end of the tube.

The modal model needed 2 minutes 10 seconds for this calculation, BEM/IM used around 20 minutes.

As a comparison to a pure plane wave model, the acoustical impedance calculated using only the plane wave mode, is shown in fig. 22 on page 19. The radiation impedance is that of a piston in an infinite baffle. The increased accuracy of the modal model, especially at higher frequencies, is evident.

Note that the impedance peaks are often somewhat higher for the MPM than for BEM and BERIM. The most probable explanation for this is that the resonance frequencies are slightly different, and that the calculation frequency coincides with the resonance frequency in one case, but not the other. There is shift in resonance frequencies that occur when modes are added, and this can be seen both from that the curves do not completely overlap, and
Figure 20: Acoustical impedance, BEM and MPM

Figure 21: Acoustical impedance, BERIM and MPM
when comparing to the plane wave calculation. One will see that the resonance frequencies are shifted somewhat away from those given by the plane wave model. As more modes are added, the results should be even more similar to the BEM/BERIM models.

### 5.2 Pressure distribution

Pressure distribution for the perfectly terminated case has been shown above. For the baffled case, fig. 23 shows a comparison between BERIM and the MPM, using 10 modes, at 8kHz. The agreement is very good. At lower frequencies, the agreement is not as good when it comes to level, especially on the axis, but the shape is still similar. See fig. 24. The reason for this is not perfectly clear, but it may be connected to the level of evanescent modes and the resolution of the BERIM mesh.

![Figure 23: Pressure distribution at z=0.11m, 8kHz](image)

![Figure 24: Pressure distribution at z=0.11m, 3kHz](image)

### 5.3 A horn example

Since the purpose of implementing the modal propagation method is to efficiently simulate loudspeaker horns, a test was also made to compare the throat impedance of an exponential horn calculated by the MPM to that calculated by BERIM, and also to the plane wave approximation often used in horn acoustics.

#### 5.3.1 Throat impedance

The exponential horn has a throat area of 10cm², a mouth area of 500cm², and a length of 30cm, corresponding to a cutoff frequency of 357Hz. For the MPM, a step size of 0.1mm and a total of 4 modes, including the plane wave mode, was used. For BERIM, a mesh size of 12 elements per wavelength at the highest frequency (8kHz) was used. The throat impedance of this horn, comparing BERIM and MPM, is shown in fig. 25.

The calculation time for the BERIM model was
Normalized Acoustic Z

Figure 22: Acoustical impedance, plane wave model

Figure 25: Acoustical impedance, BERIM and MPM
approximately 7 hours, while the MPM used a little more than 3 minutes.

As another example of the need to include the higher modes, fig. 26 on the following page shows the throat impedance of this horn calculated using only the plane wave mode, compared to using 4 modes. The higher modes are clearly needed throughout most of the operating range, to get the throat impedance correctly calculated.

5.3.2 Pressure response

The pressure response was calculated from the Rayleigh integral, as described in section 4.2. The results are shown in fig. 27 on the next page. Results for 4 modes and 15 modes are shown. It is evident that the more modes that are included, the higher up in frequency is the calculation valid. With 4 modes, the results are in good agreement up to about 3kHz, while with 15 modes the results are quite accurate up to 6kHz, and usable to 8kHz.

Run times were not directly compared, but for BERIM, using 100 frequencies, 91 field points, and the same mesh as used for the impedance calculations, the run time was about two hours. For the MPM, 15 modes at 200 frequencies and one field point was calculated in a little more than 9 minutes. These calculation times can not be directly compared to the time for the impedance calculations, as they were done on a different computer.

5.4 Discussion

While trying to obtain the same results with the two methods, it became clear that extracting modal data from the BEM model was not trivial. As shown in section 3.2, the level of the higher modes was dependent on the resolution of the mesh. It was also evident that BEM had trouble calculating the pressure close to the boundary correctly.

The modal decomposition of the BEM results could probably have been improved by using a more refined method. The method of Morgans [12] was chosen for its simplicity, but it is clear that the Least-squares approach of the pinv function will easily lead to inaccuracy in the modal amplitudes. The method used by Makarski [18] would probably be more accurate, but it is more complicated.

For this reason, modal decomposition was largely rejected, and instead direct comparisons of wave front pressure distributions and throat impedances were used.

For modeling the input impedance of the system, both methods give comparable results, as long as the higher modes also are terminated. The calculated pressure response also agree well with simulations in BERIM, as long as a sufficient number of modes are taken into account.

Using BERIM, the boundary condition at the end of the duct can be made quite similar to that used in the modal propagation method. The comparison of the two methods also show better agreement than using BEM with a $\rho c$ boundary condition.

6 Conclusion

A method for calculating the impedance and sound field in a duct of varying cross-section has been implemented. The method approximates the duct as a series of straight cylindrical tubes of varying cross-section, and both the plane wave mode and higher modes are propagated. The end of the duct is terminated by a modal impedance, which is either the characteristic impedance of the duct at that point (plane wave termination), or a radiation impedance. This radiation impedance is the impedance of a piston in an infinite baffle, with all the modes included. This method has been termed the Modal Propagation Method (MPM).

This implementation of the method has been compared to the Boundary Element Method (BEM) for the plane wave termination case, and to the Boundary Element Rayleigh Integral Method (BERIM), for baffled, radiating case. A method for decomposing the sound field from the BEM/BERIM simulations into mode amplitudes was implemented, to more easily compare the methods. However, the method gives the modal amplitudes as a least-squares fit to the actual wave front pressure distribution, and this will often produce errors in the mode amplitudes, especially if few modes are retained in the decom-
**Figure 26: Acoustical impedance, plane waves and MPM**

![Acoustical Impedance Graph](image)

**Figure 27: Pressure response for BERIM, 4 modes and 15 modes**

![Pressure Response Graph](image)
position, and/or the amplitude of the disregarded modes are significant. It also turned out that the level of the higher modes did not decrease below a certain minimum level, and this level was dependent on the mesh size.

Creating a boundary condition that absorbs higher order modes using BEM turned out to be difficult, as could be seen from how the pressure distribution at a certain surface along the tube varied as the length of the second tube was changed. It could also be seen from the differences in acoustical impedance at the input of the duct, compared to the modal model.

Since BEM did not seem to be completely reliable as a reference for the MPM, BERIM was also used as a reference. This way the boundary conditions for the two methods would be closer to each other. The results in this case were also much closer to the MPM. Especially the acoustical impedance showed excellent agreement, and also the pressure distributions agreed better than for the BEM. No modal decomposition was used on the BERIM results.

The accuracy of the MPM depends of course on the number of modes that are propagated. For this model, the level of the individual modes is more or less independent of the number of modes, but not completely. Because of the modal coupling at the discontinuities and the open end, energy will be exchanged between the modes. But this coupling can be assumed to be less critical than the “coupling” that happens in the modal decomposition routine, and fewer modes are required for convergence of levels.

But in conclusion, the modal propagation method seems to be both stable and efficient, as long as certain precautions are taken. The run times, even in Matlab, compares favorably to the run times of the BEM/BERIM code, which is compiled.

References


